

1) Assume $n \geq m$.

$(m+1)^2 - m = (m+1) + m^2$. Clearly n^2 increases faster than n , so $n^2 - m > n + m^2$ for $n > m+1$ and hence there are no solutions with $n > m+1$. It remains to consider the two cases $n = m$ and $n = m+1$.

Suppose $n = m$. Then we require that $n^2 - n$ divides $n^2 + n$. If $n > 3$, then $n^2 > 3n$, so $2(n^2 - n) > n^2 + n$. Obviously $n^2 - n < n^2 + n$, so if $n > 3$, then $n^2 - n$ cannot divide $n^2 + n$. It is easy to check that the only solutions (with $n = m$) less than 3 are $n = 2$ and $n = 3$.

Finally suppose $n = m+1$. We require $m^2 - m - 1$ divides $m^2 + 3m + 1$. If $m \geq 6$, then $m(m - 5) > 3$, so $2(m^2 - m - 1) > m^2 + 3m + 1$. Obviously $m^2 - m - 1 < m^2 + 3m + 1$, so $m^2 - m - 1$ cannot divide $m^2 + 3m + 1$ for $m \geq 6$. Checking the smaller values, we find the solutions less than 6 are $m = 1$ and $m = 2$.

Therefore the only solutions are: $(n, m) = (2, 2), (3, 3), (1, 2), (2, 1), (2, 3), (3, 2)$.

2) BZX and XYC are similar (sides parallel), so $BZ/ZX = XY/YC$. But XYZ is equilateral, so $BZ/ZY = ZY/YC$. Also $\angle BZY = \angle ZYC = 120^\circ$, so BZY and ZYC are similar. Hence $\angle ZBY = \angle YZC$. Hence YZ is tangent to the circle ZBK . Hence $YZ^2 = YK \cdot YB$

3) Let O be the center of K . Let AO meet K again at G . We show that G is the circumcenter of CEF . Note that AD is parallel to BE , so $AB = DE$. Similarly, AB is parallel to DF , so $AD = BF$. Hence the arcs AE and AF are equal. Hence the arcs GE and GF are equal, so $\angle AOE = 2\angle AGE = \angle EGF$. Hence triangles AOE and FGE are similar.

We have $CD \cdot CF = CB \cdot CE$, so $CF/CE = CB/CD = DA/DE$. Also $\angle ADE = \angle DAB = \angle FCE$. So triangles ADE and FCE are similar. Thus the figures $ADEO$ and $FCEG$ are similar. But O is the circumcenter of ADE , so G is the circumcenter of FCE .

4) Answer: 16.

We cannot have 17 terms, because then:

$$\begin{aligned} a_1 + a_2 + \dots + a_{11} &< 0 \\ a_2 + a_3 + \dots + a_{12} &< 0 \\ a_3 + a_4 + \dots + a_{13} &< 0 \\ \dots & \\ a_7 + a_8 + \dots + a_{17} &< 0 \end{aligned}$$

So if we add the inequalities we get that an expression is negative. But notice that each column is positive. Contradiction.

On the other hand, a valid sequence of 16 terms is: -5, -5, 13, -5, -5, -5, 13, -5, -5, 13, -5, -5, 13, -5, -5. Any run of 7 terms has two 13s and five -5s, so sums to 1. Any run of 11 terms has three 13s and eight -5s, so sums to -1.

5) Answer: 51.

The n th term is $a_n^3/(1 - 3a_n + 3a_n^2) = a_n^3/((1 - a_n)^3 + a_n^3) = n^3/((101 - n)^3 + n^3)$.

Hence the sum of the n th and $(101-n)$ th terms is 1.

Thus the sum from $n = 1$ to 100 is 50. The last term is 1, so the total sum is 51.

6) Assume that there is a solution. Take $m \leq n$ and the smallest possible m . Now $(36m + n)$ and $(m + 36n)$ must each be powers of 2. Hence 4 divides n and 4 divides m . So $m/2$ and $n/2$ is a smaller solution with $m/2 < m$. Contradiction.

Note that either of m or n being odd can be directly ruled out.