Problem set 2

First the questions.

1. Prove that if a, b, c are any integers, & n is an integer>3, then there exists an integer k such that none of the numbers k+a, k+b, k+c is divisible by n.

2. Consider the map F: $[0,1]X[0,1] \rightarrow [0,1]$ such that

$$F(a, x) = x + a \text{ modulo } 1$$

Note that $x_n = F^n(a, x_0)$. When a is irrational and $0<\zeta<1$, there exists non-negative integers m, n such that either $x_0 + na - m = \zeta$, in which case one and only one member of $\{x_n\}$ equals ζ, or $x_0 + na - m \neq \zeta$ for all m, n ≥ 0 and there exists an infinite subsequence $\{n_r\}$ of integers such that $0 \leq n1 < n2\ldots..$ and $x_{nr} \rightarrow \zeta$ as $r \rightarrow \infty$. Prove this result.

3. The union of disjoint knots is called a link. Find a deformation for the link shown in the picture above that interchanges the two components.

4.a) The sequence $a_1, a_2, a_3 \ldots$ of positive integers is determined by its first two members and the rule $a_{n+2} = (a_{n+1} + a_n)/\gcd(a_n, a_{n+1})$. For which values of a_1 and a_2 is it bounded?

b) p(x) is a quadratic polynomial with non-negative coefficients. Show that $p(xy)^2 \leq p(x^2)p(y^2)$.

5. Let σ be a non-singular curve of the second degree and A_1, A_2, A_3, A_4, A_5 and A_6 be points on it. Then the three points where the straight lines A_1A_5 and A_2A_4; A_3A_4 and A_1A_6; A_2A_6 and A_3A_5 meet are on the same straight line.

6. The chord CD of a circle center O is perpendicular to the diameter AB. The chord AE goes through the midpoint of the radius OC. Prove that the chord DE goes through the midpoint of the chord BC.

Now the solutions:

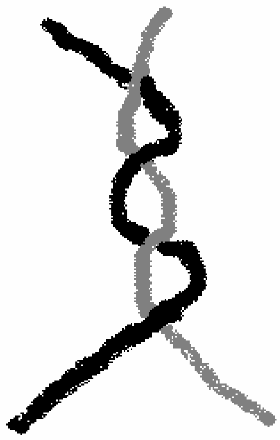
1. Let r1, r2, r3 be the remainders upon dividing the integers -a, -b, -c by n. Thus, r1, r2 and r3 are integers from the sequence 0, 1, ….. n-1 and since there are at most 3 different among the three remainders and n>3, there exists an r belonging to this sequence s.t. $r \neq r1$, $r \neq r2$, $r \neq r3$.

   If we had $n|a+r$, then in view of $-a \equiv r1 \pmod n$, we would have $n|r-r1$. However, r and r1 are integers >=0 and <n, and if their difference is divisible by n, we must have r=r1 contrary to the way r was chosen. Similarly $n \nmid b+r$ and $n \nmid c+r$. Thus the required k is r.

2. For the chosen ε, it can happen that $x_n=\varepsilon$, in which case the first part follows. If it is not possible to find such an x_n,

   Choose any integer r>0. Atleast 2 of the r+1 points $x_0, x_1, \dots x_r$ must lie in one of the r equal sub-intervals [0,1/r]….. [(r-1)/r, 1] by the pigeon hole principle. Denote these two points by x_l, x_m. For l<m, define,
   $\varepsilon=x_m-x_l$
   Therefore, $|\varepsilon|<1/r$
   $x_{m-l}=x_0+(m-l)a$ modulo 1
   $=x_0+(x_m-x_l)$ modulo 1
   $=x_0+\varepsilon$ modulo 1
   $=x_0+\varepsilon$
   Similarly, it can be shown that,
   $x_{k(m-l)}=x_0+k\varepsilon$
   Therefore there exists k such that distance from of $x_{k(m-l)}$ from ε is less that |ε|. Hence,
   $|x_{nr}-\varepsilon|<1/r$
   Where nr=k(m-l)
   Therefore we can construct a subsequence $\{x_{n1}, x_{n2}, \dots\}$ such that
   $|x_{nr}-\varepsilon|<1/r$ for r=1,2…
   Hence proved.

3. On yanking the black thread to the left, the pattern below is what one gets at the intersection of the 2 threads



One sees that the two threads are entirely symmetric at the points where they are intertwined and hence one can be converted into the other.

4. a. a1=a2=2;

Put $d_n = \gcd(a_n, a_{n+1})$. Note that d_{n+1} divides a_{n+1} and a_{n+1} and hence also $d_n a_{n+2} - a_{n+1} = a_n$. So it also divides d_n. Hence, in particular, $d_n \geq d_{n+1}$. Since all d_n are positive integers, we must have $d_n = d$ for all $n \geq$ some N.

If $d = 1$, then $a_{n+2} = a_{n+1} + a_n > a_{n+1}$ for any $n > N$. So a_n cannot be bounded.

If $d \geq 3$, then $a_{n+2} < (a_{n+1} + a_n)/2$ for all $n > N$. Hence $a_{n+2} < \max(a_{n+1}, a_n)$. Now $a_{n+3} < \max(a_{n+2}, a_{n+1}) \leq \max(a_{n+1}, a_n)$. Hence $\max(a_{n+3}, a_{n+2}) < \max(a_{n+1}, a_n)$. So we get an infinite strictly decreasing sequence of positive integers. Contradiction.

So we must have $d = 2$. Hence $a_{n+2} = (a_{n+1} + a_n)/2$ for all $n > N$. Hence $a_{n+2} - a_{n+1} = (a_n - a_{n+1})/2$. So if $a_N \neq a_{N+1}$, then for sufficiently large n we get an non-integral. Contradiction. So $a_N = a_{N+1}$. But $\gcd(a_N,a_{N+1}) = 2$, so $a_N = a_{N+1} = 2$. So $2 = (2 + aN\text{-}1)/\gcd(2,a_{N-1})$. If $\gcd(2,a_{N-1}) = 1$, then $a_{N-1} = 0$. Contradiction. So $\gcd(2,a_{N-1}) = 2$. Hence $a_{N-1} = 2 \gcd(2,a_{N-1}) - 2 = 2$. Now by a trivial induction all terms are 2. In particular a_1 and a_2 are 2.

b. Using Cauchy-Schwartz inequality, we have,

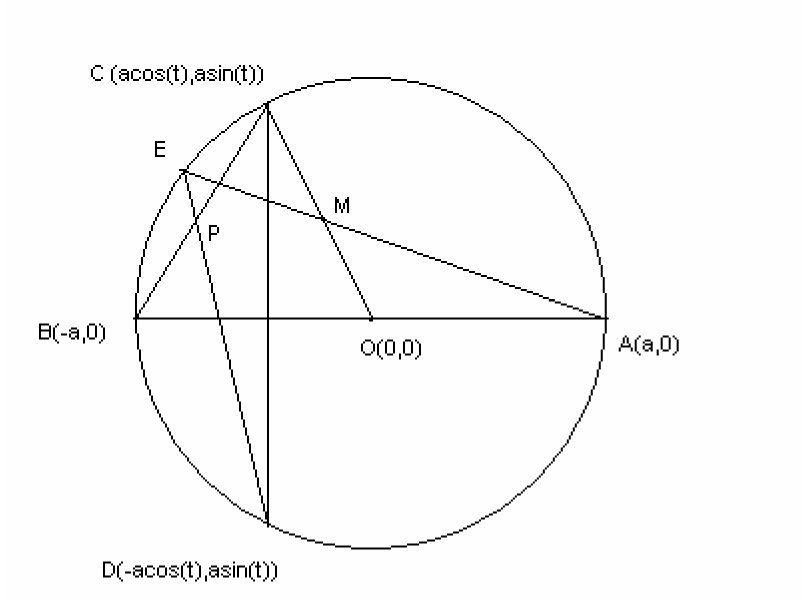
$\text{LHS} = \left((\sqrt{a}\, x^2)(\sqrt{a}\, y^2) + (\sqrt{b}\, x)(\sqrt{b}\, y) + \sqrt{c}\,\sqrt{c} \right)^2 \le (ax^4 + bx^2 + c)(ay^4 + by^2 + c)$

$= \text{RHS}$

5. Let $\sigma: x=y^2$, $\alpha_{ij}(x,y)=0$ be the equation of the straight line between A_i and A_j
   Consider $P(x,y)=\alpha_{24}\alpha_{16}\alpha_{35}-\lambda\alpha_{34}\alpha_{26}\alpha_{15}$ which is a polynomial of degree 3 in x and y.

   The co-ordinates of A_1, A_2, A_3, A_4, A_5 and A_6 satisfy $P(x,y)=0$. Take A in σ other that A_i and choose λ such that A satisfies $P(x,y)=0$. Seven points in σ satisfy $P(x,y)=0$. If we substitute y^2 for x, $P(y^2,y)$ is a polynomial of sixth degree, satisfied by seven different values of y, and hence must be an identity for every point on σ.
   $P(x,y)= (x-y^2)Q(x,y) + R(y)$; regarding $P(x,y)$ as a polynomial in x with co-efficients that depend on y.
   $P(x,y)=0$ for every point on σ.
   ⇨ $R(y)=0$ for all y.
   $P(x,y)=(x-y^2)Q(x,y)$
   $=(x-y^2)(ax+by+c)$
   Because P is of degree 3.
   Since the points under consideration do not lie on σ, they lie on $ax+by+c=0$.

6.



Let O, the center of the circle be at origin.

Let A be (a,0), then B is (-a,0).

Let C be (acost, asint), then D is (acost, -asint).

Now if E is (acosu, asinu).

Mid point of OC (say M) is (a(1+cost)/2, asin(t)/2).

E lies on AM => (Substituting E in the equation of AM we have)

2sinu = sin(u) cos(t) - sin(t) cos(u) + sin(t) ---------->1

Now we have to show that P (a(cost-1)/2, asin(t)/2) lies on line DE

⇨ to show that 2sin(t+u) = (cos(t)-1)*(sin(t)+sin(u)) – sin(t) * (cos(u)- cos(t))

⇨ to show that 4 sin(t) cos(u) + 3 sin(u) = sin (2t) --------> A

Now 1 => tan (u/2) = [2-cos(t)] / sin(t) -----------> 2

cos(a) = $1-\tan^2 a$ / $1+\tan^2 a$

⇨ from 2 we have cos(u) = [4cos(t) – 4 – cos(2t)] / [5 – 4cos(t)] -------> 3

and sin(u) = [4sint - sin(2t)] / [5-4cos(t)] ----------->4

Substituting 3 and 4 in A we have,

4 sin(t) cos(u) + 3 sin(u) = [5sin(2t) – 4sin(t)(1+cos(2t))] / (5-4cos(t)) = 2sin(t)cos(t).

Hence proved.