

## Problem Set – 1 Solutions

1) Show that  $\sqrt{x} + \sqrt{y} + \sqrt{z} \geq xy + yz + zx$  for positive reals  $x, y, z$  with sum 3

**Solution:-**

Applying AM – GM inequality on  $x^2, \sqrt{x}, \sqrt{x}$  we get

$$x^2 + \sqrt{x} + \sqrt{x} \geq 3x \quad \text{-----} \rightarrow 1$$

Adding similar inequalities for  $y, z$ , we get

$$\begin{aligned} x^2 + y^2 + z^2 + 2(\sqrt{x} + \sqrt{y} + \sqrt{z}) &\geq 3(x + y + z) \\ &= (x + y + z)^2 \quad (\text{since } x+y+z = 3) \\ &= x^2 + y^2 + z^2 + 2(xy + yz + zx). \end{aligned}$$

$$\Rightarrow (\sqrt{x} + \sqrt{y} + \sqrt{z}) \geq (xy + yz + zx)$$

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2) Show that  $\sin^{2n}x + (\sin^n x - \cos^n x)^2 \leq 1$

**Solution:-**

Put  $s = \sin x, c = \cos x$ , so  $s^2 + c^2 = 1$ .

Then LHS =  $2^n s^{2n} c^{2n} + (s^n - c^n)^2 = (2^n - 2) s^n c^n + s^{2n} + c^{2n}$  and

RHS =  $1 = (s^2 + c^2)^n = s^{2n} + c^{2n} + \sum_{i=1}^{n-1} n C_i s^{2n-2i} c^{2i}$ .

So we have to show that  $(2^n - 2) s^n c^n \leq \sum_{i=1}^{n-1} n C_i s^{2n-2i} c^{2i}$ .

Now apply AM/GM to the  $2^n - 2$  terms  $s^h c^k$ .

Note that there are the same number of terms  $s^{2n-2i} c^{2i}$  and  $s^{2i} c^{2n-2i}$  and the product of each pair is  $s^{2n} c^{2n}$

Hence the GM is  $s^n c^n$

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3) If  $a+b+c=0$ , show that  $2(ab+bc+ca)^4 = a^4 (b-c)^4 + b^4 (c-a)^4 + c^4 (a-b)^4$

**Solution:-**

If  $a+b+c=0$

Note that the following hold true

$$\begin{aligned} \Rightarrow a^2+b^2+c^2 &= -2(ab+bc+ca) \text{ -----} \rightarrow 1 \\ \Rightarrow (ab+bc+ca)^2 &= a^2b^2+b^2c^2+c^2a^2 \text{ ----} \rightarrow 2 \\ \Rightarrow (a^2+b^2+c^2)^2 &= 4 * (a^2b^2+b^2c^2+c^2a^2) \text{ -----} \rightarrow 3 \text{ (from 1 and 2)} \\ \Rightarrow a^4+b^4+c^4 &= 2 * (a^2b^2+b^2c^2+c^2a^2) \text{ -----} \rightarrow 4 \text{ (from 3)} \end{aligned}$$

Now RHS =

$$p^4+q^4+r^4 \text{ where } p = b^2-c^2, q = c^2-a^2, r = a^2-b^2$$

$$\text{now } p+q+r = 0 \text{ so from 4 and 2 we have } \text{RHS} = 2 * (pq+qr+rp)^2$$

$$\begin{aligned} \text{Substituting } p, q, r \text{ we have } \text{RHS} &= 2 * [(a^2b^2+b^2c^2+c^2a^2) - (a^4+b^4+c^4)]^2 \\ &= 2 * [(a^2b^2+b^2c^2+c^2a^2)]^2 \text{ -----} \rightarrow \text{from 4} \\ &= 2 * [(ab+bc+ca)^2]^2 \text{ -----} \rightarrow \text{from 2} \\ &= 2 * [(ab+bc+ca)^2]^2 = \text{LHS} \end{aligned}$$

4) Find  $[1/3] + [2/3] + [2^2/3] + [2^3/3] + \dots + [2^{1000}/3]$ .

Answer

$$(2/3)(2^{1000}-1)-500$$

**Solution**

$$\text{Let } f(n) = [2^n/3].$$

$$\text{Note that } f(0) = f(1) = 0.$$

$$\text{We have } 2^n = (3-1)^n = (-1)^n \pmod{3},$$

$$\text{so if } n \text{ is even, then } 2^n/3 = f(n) + 1/3 \text{ and therefore } 2^{n+1}/3 = 2f(n) + 2/3.$$

$$\text{So } f(n+1) = 2f(n).$$

$$\text{Similarly, if } n \text{ is odd, } f(n+1) = 2f(n) + 1.$$

$$\text{Thus } f(2n) = 2f(2n-1)+1 = 4f(2n-2)+1.$$

Put  $u_n = f(2n) + 1/3$ , then  $u_n = 4u_{n-1}$ . But  $u_1 = 4/3$ , so  $u_n = 4^n/3$ .

Hence  $u_1 + u_2 + \dots + u_n = (4/3)(1 + \dots + 4^{n-1}) = (4/9)(4^n - 1)$ .

So  $f(2) + f(4) + \dots + f(2n) = (4/9)(4^n - 1) - n/3$ .

We have  $f(2n+1) = f(2n)$ , so  $f(3) + f(5) + \dots + f(2n-1) = (8/9)(4^{n-1} - 1) - (2/3)(n-1)$ .

Hence  $f(2) + f(3) + \dots + f(2n) = (2/3)(4^n - 1) - n$ .

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5) The equations  $x^2 + ax + 1 = 0$  and  $x^2 + bx + c = 0$  have a common real root, and the equations  $x^2 + x + a = 0$  and  $x^2 + cx + b = 0$  have a common real root. Find  $a + b + c$ .

Answer

-3

**Solution**

Let  $p$  be the common root of  $x^2 + ax + 1 = 0$  -----→ 1

and  $x^2 + bx + c = 0$  -----→ 2

Then it must also satisfy  $(a-b)x + (1-c) = 0$  (equations 1-2)

so  $p = (c-1)/(a-b)$ .

Note that the other root of  $x^2 + ax + 1 = 0$  must be  $(a-b)/(c-1)$ , since the product of the roots is 1.

Similarly the common root of  $x^2 + x + a = 0$  and  $x^2 + cx + b = 0$  is

$x = (a-b)/(c-1)$ . -----→ 3

Therefore  $x^2 + x + a = 0$  and  $x^2 + ax + 1 = 0$  have a common root.

So it satisfies  $(a-1)x + (1-a) = 0$ . Now we cannot have  $a = 1$ , for then  $x^2 + ax + 1$  has no real roots. Hence the common root must be 1.

Hence both roots of  $x^2 + ax + 1 = 0$  are 1 and so  $a = -2$ .

So  $x^2 + bx + c = 0$  has one root 1. Then its other root must be  $c/1 = c$ . Hence  $-b = 1 + c$ , or  $b + c = -1$ . Hence  $a + b + c = -3$

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6)  $x, y$  are positive reals such that  $x + y = 2$ . Show that  $x^3 y^3 (x^3 + y^3) \leq 2$ .

**Solution:-**

Since  $x + y = 2$ ,  
 $x^3 + y^3 = (x+y)^3 - 3xy(x+y) = 8 - 6xy$

$$\Leftrightarrow x^3 y^3 (x^3 + y^3) = x^3 y^3 (8 - 6xy) = 2 * xy * xy * xy * (4 - 3xy) \rightarrow 1$$

Now using AM  $\geq$  inequality on  $xy, xy, xy, 4 - 3xy$  we have

$$\begin{aligned} x^3 y^3 (x^3 + y^3) &= 2 * xy * xy * xy * (4 - 3xy) \leq 2 * ((xy + xy + xy + 4 - 3xy) / 4)^4 \\ &= 2 * 1 \\ &= 2. \end{aligned}$$

Hence proved.