Problem set 4

The last problem set. The *ed ones are quite simple. Good luck!

1. Prove that there exists arbitrarily long arithmetic progressions formed of different integers such that every two terms of this progression are relatively prime.

2. *a. Find all real a for which there exist non-negative x_i such that

$$\sum_{k=1}^{5} k\, x_k = a$$

$$\sum_{k=1}^{5} k^3\, x_k = a^2$$

$$\sum_{k=1}^{5} k^5\, x_k = a^3$$

   *b. Prove by elementary means that if D is any integer not equal to 0, then the equation $x^2-Dy^2=z^2$ has infinitely many solutions for positive integers x, y, z such that (x, y) =1.

3. There is a group of people that is to be split into 2 groups. Each person knows at most three people. To facilitate interaction, the group is divided in such a way that each person will know at most one person in his/her own group. Show that this is possible.

4. *Consider an infinite lattice at the integer points in a two-dimensional co-ordinate space such that (x, y) is linked to (a, b) ($a \neq x$; $b \neq y$) if $|x-a| + |y-b|=3$. Show that the graph is connected, i.e. there exists a path between any 2 given points. Does there exist a cycle that contains two given points, i.e. are there 2 different paths between the same 2 points?

5. BD and CE are angle bisectors of the triangle ABC. $\angle BDE = 24^\circ$ and $\angle CED = 18^\circ$. Find angles A, B, C.

6. *Consider the sequence whose n_{th}, $(n-1)_{th}$ term and $(n-2)_{th}$ term are related as :

$$\frac{(2n+1)}{(2n-1)\cos^2\theta}\frac{T_n}{T_{n-1}} + \frac{(2n-3)\cos^2\theta}{(2n-1)}\frac{T_{n-2}}{T_{n-1}} = -2\cos 2\theta$$

and $T_1=(-1/3)\cos^3\theta \cos 3\theta$.
Consider $c=\cos^2\theta + T_1 + T_2 + \ldots\ldots\ldots + T_n + \ldots\ldots$ Prove that $\tan 2c = 2\cot^2\theta$.

Open question (Bonus points for cracking this one):

Consider a sphere that emits light in all directions from every point in its surface. Consider other spheres of different sizes, which absorb all light and reflect nothing. What is the minimum number of absorbing spheres required to ensure that the emitting sphere is invisible from afar (i.e. beyond the absorbing spheres) in every direction?